ON MANIFOLDS WITH TRIVIAL LOGARITHMIC TANGENT BUNDLE: THE NON-KÄHLER CASE.

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Abstract. We study non-Kähler manifolds with trivial logarithmic tangent bundle. We show that each such manifold arises as a fiber bundle with a compact complex parallelizable manifold as basis and a toric variety as fiber.

1. Introduction

By a classical result of Wang [11] a connected compact complex manifold X has holomorphically trivial tangent bundle if and only if there is a connected complex Lie group G and a discrete subgroup Γ such that X is biholomorphic to the quotient manifold G/Γ. In particular X is homogeneous. If X is Kähler, G must be commutative and the quotient manifold G/Γ is a compact complex torus.

Given a divisor D on a compact complex manifold $\bar{X}$, we can define the sheaf of *logarithmic differential forms*. This is a coherent sheaf. If D is locally s.n.c., this sheaf is locally free and we may ask whether it is globally free, i.e. isomorphic to $\mathcal{O}^{\dim \bar{X}}$.

In [13] we investigated this question in the case where the manifold is Kähler (or at least in class $\mathcal{C}$) and showed in particular that under these conditions there is a complex semi-torus T acting on X with $X \setminus D$ as open orbit and trivial isotropy at points in $X \setminus D$.

In this article we investigate the general case, i.e., we do not require the complex manifold X to be Kähler or to be in class $\mathcal{C}$.

There are two obvious classes of examples: First, if G is a connected complex Lie group with a discrete cocompact subgroup Γ, then $X = G/\Gamma$ is such an example with D being the empty divisor. Second, if T is a semi-torus and $T \hookrightarrow \bar{T}$ is a smooth equivariant compactification such that all the isotropy groups are again semi-tori, then $X = \bar{T}$, $D = \bar{T} \setminus T$ yields such examples ([13]).

Both classes contain many examples. E.g., every semisimple Lie group admits a discrete cocompact subgroup ([2]). On the other hand,

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every semi-torus admits a smooth equivariant compactification such that all isotropy groups are semi-tori. If the semi-torus under consideration is algebraic, i.e. a semi-abelian variety, then this condition on the isotropy groups is fulfilled for every smooth algebraic equivariant compactification.

We will show that in general every compact complex manifold $\bar{X}$ with a divisor D such that $\Omega^1(\log D)$ is trivial must be constructed out of these two special classes. First we show that there is a fibration $\bar{X} \to Y$ where the fiber is an equivariant compactification $\bar{A}$ of a semi-torus A and the base Y is biholomorphic to a quotient of a complex Lie group G by a discrete cocompact subgroup Γ. From this one can deduce that any such manifold X is a compactification of a $(\mathbb{C}^*)^l$-principal bundle over a compact complex parallelizable manifold, just like semi-tori are extension of compact complex tori by some $(\mathbb{C}^*)^l$.

2. The main results

Theorem 1. *Let $\bar{X}$ be a compact complex manifold and let D be a locally s.n.c. divisor such that $\Omega^1(\log D)$ is globally trivial (i.e. isomorphic to $\mathcal{O}_{\bar{X}}^{\dim \bar{X}}$).*

Let G denote the connected component of

$$\mathrm{Aut}(\bar{X}, D) = \{g \in \mathrm{Aut}(\bar{X}) : g(D) = D\},$$

and let $x \in X = \bar{X} \setminus D$.

Then G acts transitively on $X = \bar{X} \setminus D$ with discrete isotropy group $\Gamma = \{g \in G : g \cdot x = x\}$.

Furthermore there exists a central subgroup $C \simeq (\mathbb{C}^*)^l$ of G, a smooth equivariant compactification $C \hookrightarrow \bar{C}$, a compact complex parallelizable manifold Y and a holomorphic fibration $\pi : \bar{X} \to Y$ such that

- *π is a locally holomorphically trivial fiber bundle with $\bar{C}$ as fiber and C as structure group.*
- *The π-fibers are closures of orbits of C,*
- *The projection map π is G-equivariant and admits a G-invariant flat holomorphic connections.*
- *There is a connected complex Lie subgroup $H \subset G$ acting transitively on Y with discrete isotropy group Λ.*
- *$Lie(G) = Lie(H) \oplus Lie(C)$ and $\Gamma = \{\gamma \cdot \rho(\gamma) : \gamma \in \Lambda\}$ for some group homomorphism $\rho : \Lambda \to C$.*

Conversely, any such fiber bundle yields a log-parallelizable manifold:

Theorem 2. *Let Y be a compact complex parallelizable manifold, $C \simeq (\mathbb{C}^*)^l$, $\bar{C}$ a smooth equivariant compactification of C. Assume that all the isotropy groups of the C-action on $\bar{C}$ are reductive. Let $E_1, \ldots, E_l$*

be $\mathbb{C}^*$-principal bundles over Y, each endowed with a holomorphic connection.

Define $\bar{X}$ as the total space of the $\bar{C}$-bundle associated to the C-principal bundle ΠE_i and let D be the divisor on $\bar{X}$ induced by the divisor $\bar{C} \setminus C \subset \bar{C}$.

Then D is a locally s.n.c. divisor and $\Omega^1(\bar{X}; \log D)$ is globally trivial.

This can also be expressed in group-theoretic terms:

Theorem 3. *Let H be a complex connected Lie group, Λ a discrete cocompact subgroup, $l \in \mathbb{N}$, $C = (\mathbb{C}^*)^l$, $i : C \hookrightarrow \bar{C}$ a smooth equivariant compactification and $\rho : \Lambda \to C$ a group homomorphism. Assume that all the isotropy groups of the C-action on $\bar{C}$ are reductive.*

Define $\bar{X} = \left(H \times \bar{C}\right) / \sim$ and $X = (H \times C) / \sim$ where $(h, x) \sim (h', x')$ iff there is an element $\lambda \in \Lambda$ such that $(h', x') = (h\lambda, \rho(\lambda^{-1})x)$. Define $D = \bar{X} \setminus X$.

Then D is a locally s.n.c. divisor and $\Omega^1(\bar{X}; \log D)$ is globally trivial.

3. Logarithmic forms

Let $\bar{X}$ be a complex manifold and D a divisor. We say that D is "locally s.n.c" (where s.n.c. stands for "simple normal crossings") if if for every point $x \in \bar{X}$ there exists local coordinates $z_1, \ldots, z_n$ and a number $d \in \{0, \ldots n\}$ such that in a neighbourhood of x the divisor D equals the zero divisor of the holomorphic function $\Pi_{i=1}^d z_i$.

It is called a "divisor with only simple normal crossings as singularities" or "s.n.c. divisor" if in addition every irreducible component of D is smooth.

Let $\bar{X}$ be a compact complex manifold with a locally s.n.c. divisor D. There is a stratification as follows: $X_0 = X = \bar{X} \setminus D$, $X_1 = D \setminus \mathrm{Sing}(D)$ and for $k > 1$ the stratum X_k is the non-singular part of $\mathrm{Sing}(\bar{X}_{k-1})$. If in local coordinates D can be written as $\{z : \Pi_{i=1}^d z_i = 0\}$, then $z = (z_1, \ldots, z_n) \in X_k$ iff $\#\{i : 1 \le i \le d, z_i = 0\} = k$.

Let D be an effective divisor on a complex manifold $\bar{X}$. The sheaf $\Omega^k(\log D)$ of *logarithmic k-forms* with respect to D is defined as the sheaf of all meromorphic k-forms ω for which both $f\omega$ and $fd\omega$ are holomorphic for all $f \in \mathcal{I}_D$. If D is locally s.n.c., we may also define the algebra (with respect to exterior product) of logarithmic k-forms as the $\mathcal{O}_{\bar{X}}$-algebra generated by 1 and all df/f where f is a section $\mathcal{O}_{\bar{X}} \cap \mathcal{O}_X^*$.

Note: A logarithmic 0-form is simply a holomorphic 0-form, i.e., a holomorphic function.

For every natural number k the sheaf of logarithmic k-forms is a coherent $\mathcal{O}$-module sheaf. It is locally free if D is a locally s.n.c. divisor.

Especially, if $D = \{z_1 \cdot \ldots \cdot z_d = 0\}$, then $\Omega^1(\log D)$ is locally the free $\mathcal{O}_{\bar{X}}$-module over $dz_1/z_1, \ldots, dz_d/z_d, dz_{d+1}, \ldots, dz_n$.

For a *locally s.n.c.* divisor D on a complex manifold we define the *logarithmic tangent bundle* $T(-\log D)$ as the dual bundle of $\Omega^1(\log D)$.

Then $T(-\log D)$ can be identified with the sheaf of those holomorphic vector fields V on $\bar{X}$ which fulfill the following property:

V_x is tangent to X_k at x for every k and every $x \in X_k$.

In local coordinates: If $D = \{z : \Pi_{i=1}^d z_i = 0\}$, then $T(-\log D)$ is the locally free sheaf generated by the vector fields $z_i \frac{\partial}{\partial z_i}$ $(1 \le i \le d)$ and $\frac{\partial}{\partial z_i}$ $(d < i \le n)$.

This implies in particular (compare also [12]):

Proposition 1. *Let X be a compact complex manifold and D a locally s.n.c. divisor such that $\Omega^1(\log D)$ is globally trivial.*

Let G denote the connected component of the group $\mathrm{Aut}(X, D)$ *of all holomorphic automorphisms of X which stabilize D.*

Then the Lie algebra of G can be identified with $\Gamma(\bar{X}, \Omega^1(\log D))^*$ *and the G-orbits in $\bar{X}$ coincide with the connected components of the strata X_k.*

For every $x \in X_0 = \bar{X} \setminus D$ the isotropy group $G_x = \{g \in G : g(x) = x\}$ is discrete.

The isotropy representation of G_x on T_x is almost faithful (i.e. the kernel is discrete) for every $x \in \bar{X}$.

4. Residues

Let $\bar{X}$ be a complex manifold and D a locally s.n.c. divisor. In the preceding section we introduced the notion of logarithmic forms. Now we introduce "residues".

Let $\tau : \hat{D} \to D$ denote a normalization of D. Note that $\hat{D}$ is a (not necessarily connected) compact manifold. It is smooth, because D is locally s.n.c. Furthermore D being locally s.n.c. implies that $D' = \tau^* \operatorname{Sing}(D)$ defines a divisor on $\hat{D}$.

We are going to define

$$res : \Omega^k(X; \log D) \longrightarrow \Omega^{k-1}(\hat{D}, \log D').$$

Let $\hat{x} \in \hat{D}$ and $x = \tau(\hat{x}) \in \bar{X}$. x admits an open neighbourhood W in $\bar{X}$ with local coordinates $z_1, \ldots, z_n$ such that

(1) $z_1(x) = \ldots = z_n(x) = 0$.
(2) τ induces an isomorphism between an open neighbourhood $\hat{W}$ of $\hat{x}$ in $\hat{D}$ with $\{p \in W : z_1(p) = 0\}$.

(3) There is a number $1 \le k \le n$ such that $D|_W$ is the zero divisor of the function $z_1 \cdot \ldots \cdot z_k$.
(4) The restriction of D' to $\hat{W}$ is the pull-back of the zero divisor of the function $z_2 \cdot z_3 \cdot \ldots \cdot z_k$ on W.

Now let $\omega \in \Gamma(W, \Omega^k(\log D))$. Then

$$\omega = \eta + \frac{dz_1}{z_1} \wedge \mu$$

where η and μ have poles only along the zero set of $z_2 \cdot \ldots \cdot z_k$. We define the restriction of $res(\omega)$ to $\hat{W}$ as

$$res(\omega)|_{\hat{W}} = \tau^* \mu$$

It is easily verified that $res(\omega)$ is independent of the choice of local coordinates and the decomposition $\omega = \eta + \frac{dz_1}{z_1} \wedge \mu$ and that furthermore the only poles of $res(\omega)$ are logarithmic poles along D'.

Moreover:

Lemma 1. *Let X be a complex manifold, D a locally s.n.c. divisor and ω a logarithmic k-form ($k \in \mathbb{N} \cup \{0\}$).*

Then ω is holomorphic iff $res(\omega) = 0$. Furthermore

$$res(d\omega) = -d\, res(\omega)$$

Proof. This is easily verified by calculations in local coordinates. □

As a consequence we obtain the following result which we will need later on.

Lemma 2. *Let $\bar{X}$ be a compact complex manifold, D a locally s.n.c. divisor and ω a logarithmic 1-form.*

Then $d\omega$ is holomorphic.

Proof. The residue $res(\omega)$ is a logarithmic 0-form on the normalization $\hat{D}$ of D. But logarithmic 0-forms are simply holomorphic functions, and every holomorphic function on a $\hat{D}$ is locally constant by our compactness assumption. Therefore $dres(\omega) = 0$. Consequently $res(d\omega) = 0$, which is equivalent to the condition that $d\omega$ has no poles. □

Remark 1. *Residues can also be introduced for divisors which are not locally s.n.c., see (Saito?).*

5. Group theoretic preparations

Definition. *A (not necessarily closed) subgroup H of a topological group G is called* cocompact *iff there exists a compact subset $B \subset G$ such that $B \cdot H = \{bh : b \in B, h \in H\} = G$.*

If H is a closed subgroup of a Lie group G, then H is cocompact if and only if the quotient space G/H is compact.

Proposition 2. *Let G be a connected complex Lie group, Z its center and Γ a subgroup. Let $C = Z_G(\Gamma)^0$ denote the connected component of the centralizer $Z_G(\Gamma) = \{g \in G; g\gamma = \gamma g \ \forall \gamma \in \Gamma\}$.*

Assume that $Z\Gamma$ is cocompact.

Then $C = Z^0$.

Proof. Let B be a compact subset of G such that $BZ\Gamma = G$.

For each $c \in C$ we define a holomorphic map from G to G by taking the commutator with c:

$$\zeta_c : g \mapsto cgc^{-1}g^{-1}$$

Since c commutes with every element of $Z\Gamma$, we have $\zeta_c(G) = \zeta_c(B)$. Thus compactness of B implies that the image $\zeta_c(G) = \zeta_c(B)$ is compact. Considering the adjoint representation

$$Ad : G \to GL(Lie\, G) \subset \mathbb{C}^{n\times n}$$

we deduce that for every holomorphic function f on $GL(Lie\, G)$ the composed map $f \circ Ad \circ \zeta_c : G \to \mathbb{C}$ is a bounded holomorphic function on G and therefore constant. It follows that the image $\zeta_c(G)$ is contained in $\ker Ad = Z$.

Thus $\zeta_c(h)$ is central in G for all $c \in C$, $h \in G$. This implies

$$\zeta_c(g)\zeta_c(h^{-1}) = cgc^{-1}g^{-1}(ch^{-1}c^{-1}h) = cgc^{-1}(ch^{-1}c^{-1}h)g^{-1}$$
$$= cgh^{-1}c^{-1}hg^{-1} = \zeta_c(gh^{-1})$$

for all $c \in C$ and $g, h \in G$. Hence $\zeta_c : G \to G$ is a group homomorphism for all $c \in C$.

It follows that $\zeta_c(G)$ is a connected compact complex Lie subgroup of Z. In particular, $\zeta_c(G)$ is a compact complex torus and the commutator group G' is contained in $\ker \zeta_c$. Let A denote the smallest closed complex Lie subgroup of G containing $G'Z\Gamma$. Then A is normal, because it is a subgroup containing G', and G/A is compact, since $Z\Gamma$ is cocompact. Therefore G/A is a connected compact complex Lie group, i.e., a compact complex torus. Thus for every c the map $\zeta_c : G \to G$ fibers through G/A and is induced by some holomorphic Lie group homomorphism from G/A to T where T denotes the maximal compact complex subgroup of Z. Note that both G/A and T are compact tori. This implies that $\mathrm{Hom}(G/A, T)$ is discrete. Now $c \mapsto \zeta_c$ defines a map from the *connected* complex Lie group C to $\mathrm{Hom}(G/A, T)$. Since $\mathrm{Hom}(G/A, T)$ is discrete, this map is constant. Furthermore $\zeta_e \equiv e$. Hence $\zeta_c \equiv e$ for all $c \in C$. This implies $C \subset Z$. On the other hand the

inclusion $Z^0 \subset C$ is obvious. This completes the proof of the equality $C = Z^0$. □

Proposition 3. *Let G be a connected complex Lie group, Z its center and Γ a discrete subgroup. Assume that $Z\Gamma$ is cocompact.*

Then $Z^0\Gamma$ is closed in G.

Proof. Let $C = C_G(\Gamma)^0$ be the connected component of the centralizer. Then $C = Z^0$ be the preceding proposition. On the other hand $C\Gamma$ is closed by an argument of Raghunathan, see [12], lemma 3.2.1. □

Corollary 1. *Let G be a connected complex Lie group, Z its center and Γ a discrete subgroup. Assume that $Z\Gamma$ is cocompact.*

Then there exists a discrete cocompact subgroup Γ_1 in G with $\Gamma \subset \Gamma_1$. Moreover the commutator groups Γ', Γ'_1 can be required to be equal.

Proof. Note that Z^0 is a connected commutative Lie group. It is easy to see that there is a discrete subgroup $\Lambda \subset Z^0$ which is cocompact in Z^0 and contains $\Gamma \cap Z^0$. Define $\Gamma_1 = \Gamma \cdot \Lambda$. This is a subgroup, because Λ is central in G. By prop. 3 both $Z^0\Gamma_1$ and $Z^0\Gamma$ are closed in G. Hence we may consider the fibration

$$X = G/\Gamma_1 \to G/Z^0\Gamma_1 = G/Z^0\Gamma = Y.$$

Now Y and $Z^0/\Lambda \simeq Z^0\Gamma_1/\Gamma_1$ are both compact, hence X is likewise compact.

Finally we note that the equality $\Gamma' = \Gamma'_1$ follows from the fact that Λ is central. □

Proposition 4. *Let G be a connected complex Lie group, Z its center and Γ a discrete subgroup. Assume that $Z\Gamma$ is cocompact.*

Then $\Gamma' \cap Z$ is cocompact in $G' \cap Z$.

Proof. By cor. 1 there is a discrete cocompact subgroup $\Gamma \subset \Gamma_1 \subset G$ with $\Gamma' = \Gamma'_1$.

By passing to the universal covering there is no loss in generality in assuming that G is simply-connected.

Now $(G' \cap R)/(\Gamma_1 \cap G' \cap R)$ and $Z/(Z \cap \Gamma_1)$ are both compact, hence $(G' \cap Z)/(G' \cap Z \cap \Gamma_1)$ is compact, too. Recall that $\Gamma'_1 = \Gamma' \subset \Gamma$. By [12], prop. 3.11.2 $R \cap \Gamma'_1$ is of finite index in $G' \cap R \cap \Gamma_1$. Using $Z \subset R$, it follows that $Z \cap G' \cap \Gamma$ is of finite index in $Z \cap G' \cap \Gamma_1$. Hence $Z \cap G' \cap \Gamma$ is cocompact in $Z \cap G'$. □

Corollary 2. *Let G be a connected complex Lie group, Z its center and Γ a discrete subgroup. Assume that $Z\Gamma$ is cocompact and assume further that G acts effectively on G/Γ.*

Then $G' \cap Z$ is compact.

Proof. The subgroup $\Gamma \cap Z$ acts trivially on G/Γ, hence $\Gamma' \cap Z = \{e\}$ if G acts effectively on G/Γ. However, the trivial subgroup $\{e\}$ is cocompact in $G' \cap Z$ if and only if the latter is compact. □

6. The commutative case

Proposition 5. *Let G be a commutative connected complex Lie group and let $G \hookrightarrow \bar{X}$ be a smooth equivariant compactification, $D = \bar{X} \setminus G$.*

Then $\Omega^1(\log D)$ is a holomorphically trivial vector bundle on $\bar{X}$ if and only if the following two conditions are fulfilled:

1. *G is a semi-torus, and*
2. *every isotropy group for the G-action on $\bar{X}$ is a semi-torus.*

Proof. This is (1) $\Rightarrow$ (2) of the Main Theorem in [13]. In [13] we assumed that $\bar{X}$ is Kähler or in class $\mathcal{C}$. However, this assumption is used only in order to *deduce* that G is commutative. Hence the line of arguments in [13] still applies if instead of requiring $\bar{X}$ to be Kähler we merely assume that G is commutative. □

7. The isotropy groups

Proposition 6. *Let $\bar{X}$ be a compact complex manifold and D a locally s.n.c. divisor such that $\Omega^1(\bar{X}; \log D)$ is spanned by global sections. Let G be a connected complex Lie group acting holomorphically on $\bar{X}$ such that D is stabilized and $p \in \bar{X}$.*

Then the connected component of the isotropy group $G_p = \{g \in G : g(p) = p\}$ is central in G.

Proof. Let $w \in Lie(G)$ and $v \in Lie(G_p)$. We have to show that $[w, v] = 0$ for any such w, v. By abuse of notation we identify the elements w, v of $Lie(G)$ with the corresponding fundamental vector fields on $\bar{X}$. Now let us assume that $[w, v] \neq 0$. Then there exists a logarithmic one-form $\omega \in \Omega^1(\bar{X}; \log D)$ such that $(\omega, [w, v])$ is not identically zero. Since G stabilizes D, the vector fields w, v are tangent to D and thus sections in $T(-logD)$ which is dual to $\Omega^1(\log D)$. Therefore (ω, w) and (ω, v) are global holomorphic functions on $\bar{X}$. Because $\bar{X}$ is compact, they are constant. Hence $w(\omega, v)$ and $v(\omega, w)$ vanish both. Therefore

$$\omega([w, v]) = d\omega(w, v).$$

Now $d\omega$ has no poles (lemma 2) and v vanishes at one point, namely p. Thus $\omega([w, v])$ is a global holomorphic function on a compact connected manifold which vanishes at one point, i.e., $\omega([w, v]) \equiv 0$ contrary to our assumption on ω. We have thus deduced that the assumption $[w, v] \neq 0$ leads to a contradiction. Hence $[w, v] = 0$ for all $w \in Lie(G)$ and all $v \in Lie(G_p)$. It follows that G_p^0 is central in G. □

8. Cocompactness of $Z\Gamma$

We fix some notation to be used throughout this section:

- $\bar{X}$ is a compact complex manifold,
- D a loc. s.n.c. diuvisor on $\bar{X}$ and $X = \bar{X} \setminus |D|$,
- We assume that $\Omega^1(\bar{X}, \log D$ is a globally trivial $\mathcal{O}_{\bar{X}}$-module.
- $\mathrm{Aut}(\bar{X}, D) = \{g \in \mathrm{Aut}(\bar{X}) : g(D) = D\}$,
- G denotes the connected component of $\mathrm{Aut}(\bar{X}, D)$ which contains the identity map.
- x denotes a fixed point in $X = \bar{X} \setminus |D|$,
- Γ denotes the isotropy group at x, i.e., $\Gamma = \{g \in G : g \cdot x = x\}$
- Z denotes the center of G.

Proposition 7. *$Z\Gamma$ is cocompact in G.*

Proof. Equip $\bar{X}$ with some Riemannian metric. Let $p \in |D|$. Assume that $p \in D_1 \cap \ldots D_j$ and $p \notin D_i$ for $i > j$. Then there are logarithmic vector fields v_i and local holomorphic cooordinates z_i near p such that $v_i = z_i \frac{\partial}{\partial z_i}$ for $1 \leq i \leq j$. As a consequence, for every point $p \in \bar{X}$ there exists an open neighborhood W_p and a positive number $\epsilon_p > 0$ such that the following assertion holds: *For every $y \in W_p \setminus |D|$ there exists an element $g \in G_p^0$ such that $d(g \cdot w, |D|) > \epsilon_p$.* Since $\bar{X}$ is compact, it can be covered by finitely many of the open sets W_p. We choose such a finite collection and define ϵ as the minimum of the ϵ_p. Let H be the subgroup of G generated by all the subgroups G_p^0.

Then we have: *For every $y \in X \setminus |D|$ there exists an element $g \in H$ such that $d(g \cdot y, |D|) > \epsilon$.*

Compactness of X implies that $C = \{z \in X : d(z, |D|) \geq \epsilon\}$ is compact. We have $H \cdot C = X \setminus |D|$. We can choose a compact subset $K \subset G$ such that the natural projection from G onto $G/\Gamma \simeq X \setminus |D|$ maps K onto C. Then $G = H \cdot K \cdot \Gamma$. Observe that H is central in G due to prop 6. Hence

$$G = H \cdot K \cdot \Gamma = K \cdot H \cdot \Gamma$$

and $H \subset Z$. It follows that $Z\Gamma$ is cocompact in G. $\square$

Corollary 3. *$Z^0\Gamma$ is closed and cocompact in G.*

Proof. Implied by prop. 3 and 7. $\square$

Proposition 8. *$G/Z^0\Gamma$ is a compact parallelizable manifold and the natural projection $G/\Gamma \to G/Z^0\Gamma$ extends to a holomorphic map from $\bar{X}$ onto $G/Z^0\Gamma$.*

Proof. $G/Z^0\Gamma$ can be described as the quotient of the complex Lie group G/Z^0 by the discrete subgroup $\Gamma/(\Gamma \cap Z^0)$ and is therefore a compact parallelizable manifold.

Now let p be a point in $D \setminus \mathrm{Sing}(D)$. Choose local cooordinates near p such that $D = \{z_1 = 0\}$. Since $\Omega^1(\log D)$ is trivial, there is a global vector field v such that

$$(\frac{1}{z_1} v)_p = \frac{\partial}{\partial z_1}_p.$$

Now v is central in $Lie(G)$ and G acts locally transitively on $D \setminus \mathrm{Sing}(D)$. Therefore v vanishes as holomorphic vector field on D. It follows that $w = \frac{1}{z_1} v$ is a holomorphic vector field near p. Moreover $w_p = \frac{\partial}{\partial z_1}_p$. Now w, as any vector field, is locally integrable. Therefore there is a local coordinate system near p in which w equals $\frac{\partial}{\partial z_1}$. Since v is central as element in $Lie(G)$, v, as a holomorphic vector field on $X \setminus D$, is tangent to the fibers of $\tau : G/\Gamma \to G/Z^0\Gamma$. Thus in this local coordinate system near p, the map τ depends only on the variables $z_2, \ldots, z_n$. But this implies that τ, which is defined on $\{z_1 \neq 0\}$, extends to a map defined on the whole neighbourhood of $p \in D$. In this way we say that τ, originally defined on $X \setminus D$, extends to a holomorphic map defined on $X \setminus \mathrm{Sing}(D)$.

Next we observe that the image space $G/Z^0\Gamma$ is complex parallizable, and therefore has a Stein universal covering. It follows that we can use the classical Hartogs theorem to extend τ through the set $\mathrm{Sing}(D)$ which has codimension at least two in X. □

Proposition 9. *Let Y be a complex manifold on which G acts transitively, and $\pi : X \to Y$ an equivariant holomorphic map.*

Then each fiber F of π is smooth, $D \cap F$ is a s.n.c. divisor in F and $\Omega^1(F, \log(D \cap F))$ is trivial.

Proof. A generic fiber is smooth. Since π is equivariant and G acts transitively on Y, it follows that every fiber is smooth. Let $S = \{y \in Y : \pi^{-1}(y) \subset D\}$. Then S is G-invariant. Hence either $\pi(S) = Y$ or $S = \emptyset$. But $\pi(S) = Y$ would imply $D = X$ contrary to D being a divisor. Hence $S = \emptyset$, i.e. D does not contain any fiber of $\pi : X \to Y$.

Now we consider the usual stratification of D (as described in §3). At each point $p \in D_k$ the divisor D can locally be defined as $D = \{f_1 \ldots f_k = 0\}$ where $df_1, \ldots, df_k$ are linearly independent in T_p^*X. Since D can not be a pull-back of a divisor on Y, for a generic choice of p there will be local functions $f_1, \ldots, f_k$ near p such that $D = \{f_1 \cdot \ldots \cdot f_k = 0\}$ and in addition such that the df_i are linearly independent as elements in T_pF (with $F = \pi^{-1}(\pi(p))$). In the same spirit as in the

definition of S above we can now define the set of points in D where this fails. Observing that these sets are invariant, and recalling that G acts transitively on the connected components of the strata D_k, we deduce that they must empty. Thus $D \cap F$ is always a s.n.c. divisor.

By similar arguments we see that $\Omega^1(F, \log(D \cap F))$ is trivial. $\square$

9. Proofs for the theorems

Proof of theorem 1. Let Z denote the center of G. By definition G is a subgroup of the automorphism group of $\bar{X}$ and therefore acts effectively on $\bar{X}$. By cor. 3 $Z^0\Gamma$ is closed in G and due to prop. 8 there is a holomorphic fiber bundle $\pi : \bar{X} \to Y_0 = G/Z^0\Gamma$ which extends the natural projection $G/\Gamma \to Y_0 = G/Z^0\Gamma$. From prop 9 we deduce that a typical fiber F of $\bar{\pi}$ is log-parallelizable with respect to the divisor $F \cap D$. Since Z is commutative, this implies (due to prop. 5) that Z^0 is a semi-torus. Let C be a maximal connected linear subgroup of Z^0. Then $C \simeq (\mathbb{C}^*)^l$ for some $l \in \mathbb{N}$ and Z^0/C is a compact complex torus. The fibration $Z^0 \mapsto Z^0/C$ induces a tower of fibrations

$$G/\Gamma \to G/C\Gamma \to G/Z^0\Gamma$$

which (using prop. 8) extends to fibrations

$$\bar{X} \to Y = G/C\Gamma \to G/Z^0\Gamma$$

Now Y is parallelizable because C is normal in G and compact because both $G/Z^0\Gamma$ and Z^0/C are compact.

From cor. 2 we deduce that $G' \cap Z$ is compact. Since $C \subset Z$ and $C \simeq (\mathbb{C}^*)^l$, it follows that $G' \cap C$ is discrete. Thus we can choose a complex vector subspace $V \subset Lie(G)$ such that $Lie(G') \subset V$ and $Lie(G) = V \oplus Lie(C)$. The condition $Lie(G') \subset V$ implies that V is an ideal in $Lie(G)$. Hence V is the Lie algebra of a normal Lie subgroup H of G. Furthermore $H \cdot C = G$ due to $Lie(G) = V \oplus Lie(C)$. The condition $H \cdot C = G$ implies that H acts transitively on $Y = G/C\Gamma$. For dimension reasons the isotropy group $\Lambda = H \cap (C\Gamma)$ is discrete.

Because G acts effectively on X, the intersection $Z \cap \Gamma$ must be trivial. Hence $C \cap \Gamma = \{e\}$. It follows that the projection map $\tau : G \to G/C \simeq H/(H \cap C)$ maps Γ injectively onto $\tau(\Gamma) = C\Gamma/C$.

We claim that $\tau(\Gamma) = \tau(\Lambda)$. Indeed, $\Lambda = H \cap (C\Gamma) \subset C\Gamma$ implies $\tau(\Lambda) \subset \tau(\Gamma)$. On the other hand, if $\gamma \in \Gamma$, then there exists an element $c \in C$ such that $\gamma c \in H$, because $H \cdot C = G$. Then $\tau(\gamma) = \tau(\gamma c)$ and $\gamma c = c\gamma \in H \cap (C\Gamma) = \Lambda$. Hence $\tau(\Gamma) \subset \tau(\Lambda)$.

It follows that there exists a map $\rho : \Lambda \to C$ such that for each $\lambda \in \Lambda$ the product $\lambda\rho(\lambda)$ is the unique element γ of Γ with $\tau(\lambda) = \tau(\gamma)$.

One verifies easily that ρ is a group homomorphism, using the facts that τ is a group homomorphism and that C is central.

To obtain the G-invariant flat connection, we observe that the fundamental vector fields of $Lie(H) = V$ induce a decomposiition $T_x\bar{X} = V \oplus \ker(d\pi)$ in each point $x \in \bar{X}$. This connection is obviously G-invariant. Moreover it is flat, because $V = Lie(H)$ is a *Lie subalgebra* of $Lie(G)$.

This completes the proof. □

Proof of theorem 2. Due to prop. 5 we know that $\bar{C}$ is log-parallelizable, i.e., that $\Omega^1(\bar{C}, \log(\bar{C} \setminus C))$ is isomorphic to $\mathcal{O}^l_{\bar{C}}$. Moreover, this isomorphism is given by meromorphic 1-forms which are dual to a basis of the C-fundamental vector fields. Let $V_1, \ldots, V_l$ be such a basis of C-fundamental vector fields and $\eta_1, \ldots, \eta_l$ a dual basis for the logarithmic one-forms on $\bar{C}$. We may regard $\bar{C}$ as one fiber of the projection map from $\bar{X}$ onto Y. Using the C-principal right action of C we extend $V_1, \ldots, V_l$ to holomorphic C-fundamental vector fields on all of $\bar{X}$. Let $H \subset T\bar{X}$ be the horizontal subbundle defined by the connection. Then we can extend the meromorphic one-forms η_i as follows: For each i we require that $\eta_i(Y_j) = \delta_{ij}$ and that η_i vanishes on H.

Let ω_1, ω_r be a family of holomorphic 1-forms on Y which gives a trivialization of the tangent bundle TY. Define $\mu_i = \pi^*\omega_i \in \Omega^1(\bar{X})$.

We claim that the family $(\mu_i)_i$ together with the family η_i gives a trivialization of the sheaf of logarithmic one-forms on $\bar{X}$.

Evidently the μ_i are holomorphic. Since the restriction of η_i to a fiber is logarithmic and $\eta_i|_H \equiv 0$, it is clear that $f\eta_i$ is holomorphic for any locally given function f vanishing on D. It remains to show that $fd\eta_i$ is holomorphic as well. To see this, we calculate $fd\eta_i(Y, Z)$ for holomorphic vector fields Y, Z. Recall that

$$d\eta_i(Y, Z) = Y\eta_i Z - Z\eta_i Y - \eta_i([Y, Z])$$

It suffices to verify holomorphicity for a base of vector fields. Thus we may assume that each of the vector fields Y and Z is horizontal or vertical (with respect to the connection). If both are vertical, there is no problem since η_i restricted to a fiber is logarithmic. If both are horizontal, then $\eta_i Y = \eta_i Z = 0$. Furthermore $[Y, Z]$ is horizontal, because the connection is flat. Hence $\eta_i([Y, Z]) = 0$ and consequently $d\eta_i(Y, Z) = 0$. Finally let us discuss the case where Y is horizontal and Z is vertical. Then $\eta_i Y = 0$ and moreover $[Y, Z] = 0$ because H is defined by a connection for the C-bundle and therefore C-invariant. Thus $fd\eta_i(Y, Z) = fY\eta_i Z$. Since the fiber $\bar{C}$ is log-parallelizable, it

suffices to consider the case where, up to multiplication by a meromorphic function, Z agrees with a C-fundamental vector field V. Let ϕ be a defining function for the zero locus of V. We may then assume that $Z = \frac{1}{\phi}V$. Then

$$fd\eta_i(Y,Z) = fYd\eta_i\frac{1}{\phi}V = -f\frac{Y\phi}{\phi^2}(\eta_iV) + \frac{f}{\phi}Y(\eta_iV)$$

Since η_i is dual to V_i, the function η_iV is constant and the second term vanishes. Therefore

$$fd\eta_i(Y,Z) = -\left(\frac{f}{\phi}\right)\left(\frac{Y\phi}{\phi}\right)(\eta_iV)\,.$$

We claim that all three factors are holomorphic. Indeed, the zero locus of ϕ is contained in D (with multiplicity one) and f vanishes on D. By construction the horizontal vector field V is tangent to D, hence $V(\phi)$ vanishes along the zero locus of ϕ (which is contained in D) and therefore $V\phi/\phi$ is holomorphic. Finally η_iV is evidently holomorphic, since it is a constant function. □

Proof of theorem 3. The obvious connection on the trivial bundle $H \times C \to H$ induces a flat connection on $H \times (\bar{C}/\sim) \to Y = H/\Lambda$.

Hence the statement follows from the preceding theorem (thm. 2). □

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